Applicability of Three-parameter Weibull Distribution to Multiaxial Fracture of Ceramics

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Abstract

This paper addresses problems on the application of three-parameter Weibull distribution to multi-axial stress states. The problems are of practical importance since the fracture strength distribution of engineering ceramics is better characterized by three-parameter Weibull distribution than by two-parameter one. Based on a model, it is shown theoretically that (a) three-parameter Weibull distribution can be derived for multi-axial stress states, and (b) Weibull modulus is dependent on stress states. Specifically, Weibull modulus, m, for uniaxial tension is replaced by m-1/2 and m-1 for equi-biaxial and equi-triaxial stress states, respectively.

1. Introduction

Investigations into the fracture strength distribution of ceramic materials under multi-axial stress states is important from the viewpoint of reliability. The author[1)] demonstrated theoretically that the 2-parameter Weibull distribution can be extended to multi-axial stress states, and that Weibull modulus is independent of stress states. The present study examines the fracture statistics using the 3-parameter Weibull distribution, i.e., the problems in relation to the 3-parameter Weibull distribution extended to multi-axial stress states and the dependence of Weibull modulus on stress states. These problems are of practical importance since the fracture strength distribution of recent engineering ceramics is better characterized by 3-parameter Weibull distribution than by 2-parameter one.

Vardar and Finnie[2)] argued that the 3-parameter Weibull distribution can not be applied theoretically to multi-axial stress states. Other investigators also showed implicitly similar results [3)-5)]. The analysis in the present paper will reveal if these results are valid.

2. Theoretical Analysis

Suppose the population of failure strengths of ceramic specimens in uni-axial tension follows a 3-parameter Weibull distribution

$$F(\sigma) = 1-\exp[-(\frac{\sigma-\sigma_u}{\beta})^m] \qquad (1)$$

where m, β and σ_u are shape, scale and location parameters, respectively. We now study whether or not the same type of distribution (3-parameter Weibull distribution) can be derived theoretically for multi-axial stress states. Following assumptions are made.

(1) Each specimen contains a large number of flaws, and the weakest link theory is applicable.

(2) These flaws are modelled as internal circular cracks with their normal being oriented uniformly in all the directions.

(3) Unstable crack extension is governed by the criterion of

$K_I \geqq K_c$, where K_I is the mode I stress intensity factor, and K_c is the fracture toughness.

Consider an internal circular crack as shown in Fig.1, where σ_1, σ_2 and σ_3 are principal stresses. The normal stress σ_n acting on the crack plane is expressed by

$$\sigma_n = \sigma_1 \sin^2\theta \cos^2\psi + \sigma_2 \sin^2\theta \sin^2\psi + \sigma_3 \cos^2\theta \qquad (2)$$

K_I is given by $K_I = (2/\pi)\sigma_n\sqrt{\pi a}$, where a is the crack radius. The assumption (3) for unstable crack extension is rewritten by

$$\sigma_n \geqq \sigma_{cr} \qquad (3)$$

where σ_{cr} is given by

$$\sigma_{cr} = \frac{\pi}{2} \cdot \frac{K_c}{\sqrt{\pi a}} \qquad (4)$$

Let $G(\sigma_{cr})$ be the distribution function of σ_{cr}. When the left-hand tail of $G(\sigma_{cr})$ has the form of

$$G(\sigma_{cr}) = \left(\frac{\sigma_{cr} - \sigma_u}{\beta_1} \right)^{m-1} \qquad (5)$$

the fracture strength distribution in uniaxial tension follows Eq.(1) as shown below. Here, m and σ_u in Eq.(5) are the same parameters as those in Eq.(1), but β_1 is different from β. β_1 stands for a kind of strength of a crack whereas β stands for a kind of strength of a specimen containing a large number of cracks. Now, Eq.(1) will be derived from Eq.(5). Suppose that a fictitious specimen containing only one crack is subjected to uniaxial tension of $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, \sigma)$. Let $F_1(\sigma)$ be the distribution function of the fracture strength of this fictitious specimen. Noting that σ_n is given by $\sigma_n = \sigma\cos^2\theta$, and that the orientational probability of the crack within $\theta \sim \theta + d\theta$ and $\psi \sim \psi + d\psi$ is equal to $\sin\theta d\theta d\psi / 4\pi$, $F_1(\sigma)$ is expressed by

$$F_1(\sigma) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} G(\sigma\cos^2\theta) \sin\theta d\theta d\psi \qquad (6)$$

Here, it is to be understood that G=0 for $\sigma\cos^2\theta<\sigma_u$. Substituting Eq.(5) into Eq.(6), the left-hand tail of $F_1(\sigma)$ is expressed by

$$F_1(\sigma) = \frac{1}{4\pi}\iint_A \left(\frac{\sigma\cos^2\theta-\sigma_u}{\beta_1}\right)^{m-1}\sin\theta d\theta d\psi \tag{7}$$

where A is the domain of $\sigma\cos^2\theta\geqq\sigma_u$. Integrating Eq.(7) and noting that the left-hand tail of $F_1(\sigma)$ is concerned yield

$$F_1(\sigma) = \frac{1}{2m}\cdot\frac{(\sigma-\sigma_u)^m}{\beta_1{}^{m-1}\sigma_u} \tag{8}$$

Let $F(\sigma)$ be the distribution function of the fracture strength of real specimens. It is given by the asymptotic distribution resulting from $F_1(\sigma)$, since real specimens contain a large number of cracks by the assumption (1). This asymptotic distribution is governed by the left-hand tail of $F_1(\sigma)$. Based on the statistics of extremes[6], $F(\sigma)$ follows the 3-parameter Weibull distribution of Eq.(1) when the left-hand tail of $F_1(\sigma)$ has the form of Eq.(8). Thus, it was demonstrated that when the left-hand tail of $G(\sigma_{cr})$ has the form of Eq.(5), $F(\sigma)$ follows Eq.(1). Using Eq.(5), we next derive the fracture strength distribution for multi-axial stress states. The case of equi-biaxial and of equi-triaxial tensions are discussed.

(a) Equi-biaxial tension

Consider the equi-biaxial stress state of $(\sigma_1,\sigma_2,\sigma_3)=(\sigma,\sigma,0)$. Noting that σ_n is now given by $\sigma_n=\sigma\sin^2\theta$, the distribution function of the fracture strength of fictitious specimens containing only one crack, $F_1(\sigma)$, is

$$F_1(\sigma) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} G(\sigma\sin^2\theta)\sin\theta d\theta d\psi \tag{9}$$

Substituting Eq.(5) into Eq.(9), the left-hand tail of $F_1(\sigma)$ is expressed by

$$F_1(\sigma) = \frac{1}{4\pi}\iint_A \left(\frac{\sigma\sin^2\theta-\sigma_u}{\beta_1}\right)^{m-1}\sin\theta d\theta d\psi \qquad (10)$$

and then becomes

$$F_1(\sigma) = \frac{1}{2}B(m,\frac{1}{2})\frac{(\sigma-\sigma_u)^{m-1/2}}{\beta_1{}^{m-1}\,\sigma_u{}^{1/2}} \qquad (11)$$

where $B(m,1/2)$ is the beta function. The fracture strength distribution of real specimens each containing a large number of flaws, $F(\sigma)$, can be obtained using Eq.(11) as follows.

$$F(\sigma) = 1-\exp[-(\frac{\sigma-\sigma_u}{\beta'})^{m-1/2}] \qquad (12)$$

Eq.(12) is a 3-parameter Weibull distribution with Weibull modulus of $m-1/2$. β' in Eq.(12) is different from β in Eq.(1) since β' is for the equi-biaxial stress state whereas β is for the uniaxial stress state.

(b) Equi-triaxial tension

Consider the equi-triaxial stress state of $(\sigma_1,\sigma_2,\sigma_3)=(\sigma,\sigma,\sigma)$. Noting that σ_n is given by $\sigma_n=\sigma$, the left-hand tail of the distribution function of the fracture strength of the fictitious specimens containing only one crack, $F_1(\sigma)$, is provided by

$$F_1(\sigma) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\sigma-\sigma_u}{\beta_1}\right)^{m-1}\sin\theta d\theta d\psi$$

$$= \left(\frac{\sigma-\sigma_u}{\beta_1}\right)^{m-1} \qquad (13)$$

Hence, the fracture strength distribution of real specimens, $F(\sigma)$, is

$$F(\sigma) = 1-\exp[-(\frac{\sigma-\sigma_u}{\beta''})^{m-1}] \qquad (14)$$

This is a 3-parameter Weibull distribution with Weibull modulus of $m-1$.

β'' is different from β and β' due to difference in stress states.

In the preceding analyses, the criterion of $K_I \geqq K_c$ was employed for unstable crack extension. We conducted a similar analysis using the energy release rate criterion of $(1-\nu^2)(K_I^2+K_{II}^2)/E+(1+\nu)K_{III}^2/E \geqq G_c$, where E and ν are Young's modulus and Poisson's ratio, respectively. This criterion is based on the assumption that the crack extends along its own plane. Assuming $\nu=0$ for simplicity of calculation, this criterion is expressed alternatively by

$$\sigma_n^2 + \tau_n^2 \geqq \frac{\pi E G_c}{4a} \qquad (15)$$

where τ_n is the resultant shear stress acting on the crack plane. It was derived in the present work that Eq.(15) instead of Eq.(3) leads to the same conclusions as obtained above. The stress state dependence of Weibull modulus shown above is rather significant in practice, since Weibull modulus, m, of ceramic materials appearing in the 3-parameter Weibull distribution is appreciably small (say $m \approx 3$).

3. Discussion

As shown in the previous Chapter, 3-parameter Weibull distribution functions can be derived for equi-biaxial and equi-triaxial stress states when the same type of distribution function is assumed for uniaxial stress state. This result is in contrast with the conclusion of Vardar and Finnie[2]. This discrepancy stems from the following reason. Vardar and Finnie assumed that Eq.(1) and the criterion of $\sigma_n \geqq \sigma_{cr}$ hold for uni-axial tension. They thought that for a multi-axial stress state $(\sigma_1, \sigma_2, \sigma_3)$, Eq.(1) is modified by

$$F(\sigma_1,\sigma_2,\sigma_3) = 1-\exp[-\frac{k}{4\pi}\iint_A(\sigma_1\sin^2\theta\cos^2\psi+\sigma_2\sin^2\theta\sin^2\psi$$

$$+\sigma_3\cos^2\theta-\sigma_u)^m\sin\theta d\theta d\psi] \quad (16)$$

where the factor k is defined by $k=2(2m+1)/\beta^m$ so that $F(\sigma_1,\sigma_2,\sigma_3)$ coincides with Eq.(1) when $(\sigma_1,\sigma_2,\sigma_3)=(\sigma,0,0)$ and $\sigma_u=0$. Taking $\sigma_1=\sigma$ and $\sigma_2=\sigma_3=0$, Eq.(16) becomes

$$F(\sigma) = 1-\exp[-\frac{k}{4\pi}\iint_A(\sigma\sin^2\theta\cos^2\psi-\sigma_u)^m\sin\theta d\theta d\psi] \quad (17)$$

Since the integration in Eq.(17) does not have the form proportional to $(\sigma-\sigma_u)^m$, Eq.(16) never reduces to Eq.(1). Thus, Vardar and Finnie concluded that the 3-parameter Weibull distribution function cannot be applied to multi-axial stress states.

The argument of Vardar and Finnie involves a crucial jump in logic from Eq.(1) to Eq.(16). Eq.(16) seems correct apparently, but actually it does not result from Eq.(1). The fracture strength distribution for multi-axial stress states should be derived using the statistics of extremes and hence by the left-hand tail of the distribution of σ_{cr}. If this is done, 3-parameter Weibull distributions for multi-axial stress are derived as addressed in Chapter 2.

4. Conclusions

(1) When the fracture strength of a ceramic material in uniaxial tension follows a 3-parameter Weibull distribution, the same type of distribution can be derived theoretically for equi-biaxial and equi-triaxial stress states. This suggests that 3-parameter Weibull distribution can be applied to any multi-axial stress states.

(2) The modulus of a 3-parameter Weibull distribution is dependent on stress states. Specifically, Weibull modulus, m, for uniaxial tension, must be replaced by m-1/2 and m-1 for equi-biaxial and equi-triaxial stress states respectively. The stress state dependence is of practical importance, since Weibull modulus, m, of usual engineering ceramics appearing in a 3-parameter Weibull distribution is rather small (say $m \approx 3$).

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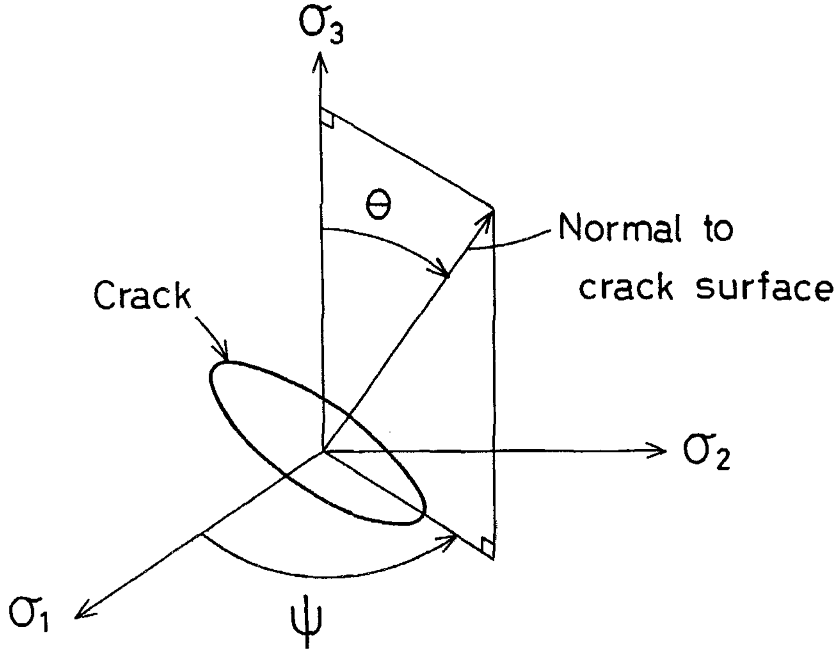

Fig.1 Internal circular crack in a multi-axial stress state.