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A discrete model of one-dimensional elastic media and the soliton theory

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A discrete model of an extensible string in three dimensional space is proposed in this paper. The model contains the bending and twisting, and becomes the special Cosserat elastic string in the continuous limit. We also present a new method of analyzing a string by the soliton theory, which can reduce the basic equations to a simpler tractable form. The discrete basic equations are also shown to be suitable for numerical simulations of string dynamics.

Key words: Soliton, elastica, Cosserat string, numerical sumulation

1. Introduction

In this paper, we propose a discrete model of an extensible string in three-dimensional space, which contains the bending and twisting of a string. Moreover, it is shown that the new model produces the special Cosserat string in the continuous limit. We point out that the soliton theory will be quite effective when we study one-dimensional media. The reason is that the Frenet-Serret formula can be considered as the Lax pair in the soliton theory. Goldstein and Petrich[1] have shown that the dynamics of a curve in a plane is governed geometrically by the modified Korteveg-de Vries(mKdV) equation in a particular case. A mathematical curve, however, is different from a physical string, which has finite cross section. Thus in the previous papers[2,3] we have proposed a way of applying these soliton approaches to analyze dynamics of physically continuous and discrete strings in a plane. We use this method in this paper and consider dynamics of a string in three-dimensional space.

2. Elastic string and Cosserat theory

Let us consider an extensible string in the three dimensional space. The axis of the string is parametrized by σ, which represents the unstretched length of the string. It forms a space curve with position $\mathbf{r} = \mathbf{r}(\sigma, t)$. The unit tangent vector $\mathbf{t}$ for the axial curve is given by the relation

$$\frac{\partial \mathbf{r}}{\partial \sigma} = \sqrt{g}\,\mathbf{t}, \tag{1}$$

where g is a metric which represents the stretch of the axis defined by

$$g = \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \sigma}. \tag{2}$$

Using g, the arclength s of the string is given by

$$s = \int_0^\sigma \sqrt{g(\sigma', t)}\, d\sigma'. \tag{3}$$

One can define a local orthonormal basis $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ by

$$\frac{\partial}{\partial s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & -\eta \\ -\kappa & 0 & \tau \\ \eta & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{4}$$

where η and κ are called components of curvature and τ the twist.

The conservation of linear and angular momentum leads to equations for the force and the moment which take the form

$$\rho Z \frac{d^2 \mathbf{r}}{dt^2} = \frac{\partial \mathbf{S}}{\partial \sigma} \tag{5}$$

$$\rho I(\mathbf{n} \times \frac{d^2 \mathbf{n}}{dt^2} + \mathbf{b} \times \frac{d^2 \mathbf{b}}{dt^2}) = \frac{\partial \mathbf{M}}{\partial \sigma} + \frac{\partial \mathbf{r}}{\partial \sigma} \times \mathbf{S} \tag{6}$$

where $\mathbf{S}$ and $\mathbf{M}$ are the resultant stress and moment, ρ the line density, Z the area of a cross section and I the geometrical moment of inertia of the string. The total derivative with respect to time is written as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{ds}{dt}\frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \left(\int \frac{d}{dt} \ln \sqrt{g} ds\right) \frac{\partial}{\partial s}. \tag{7}$$

It is noted that $\partial/\partial\sigma = \sqrt{g}\partial/\partial s$ and the derivative with respect to t does not commute with the derivative with respect to s but with σ because of the stretching effect. In (5) and (6) the stress and moment are represented by

$$\mathbf{S} = F\mathbf{t} + Q_1\mathbf{n} + Q_2\mathbf{b}, \tag{8}$$

$$\mathbf{M} = M_1\mathbf{t} + M_2\mathbf{n} + M_3\mathbf{b}, \tag{9}$$

where F and M_i are given by linear constitutive equations:

$$F = EZ(\sqrt{g}-1), \quad (10)$$
$$M_1 = GJ\tau\sqrt{g}, \quad (11)$$
$$M_2 = EI\eta\sqrt{g}, \quad (12)$$
$$M_3 = EI\kappa\sqrt{g}, \quad (13)$$

where E is the modulus of elasticity and G is the shear modulus and $J = 2I$ because of neglecting the warping. The resultant shears Q_1 and Q_2 are not determined by constitutive equations and considered as reactive parameters in the equations of balance of momentum.

3. Continuous soliton theory

We present a way of the analyzing of the string in three-dimensional space by the soliton theory. Let the curve dynamics be of the form

$$\frac{d\mathbf{r}}{dt} = U\mathbf{t} + W\mathbf{n} + V\mathbf{b} \quad (14)$$

and

$$\frac{d}{dt}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & A & B\\ -A & 0 & C\\ -B & -C & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} \quad (15)$$

without loss of generality, where U, W and V are components of a velocity of the string and A, B and C represent dynamics of a basis which are determined below. Differentiating (2) with respect to time and using (14) we obtain

$$\frac{d}{dt}g = 2g(\frac{\partial U}{\partial s} - \kappa W + \eta V). \quad (16)$$

From (3) and (16), we have the commutation

$$\frac{\partial}{\partial s}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial s} = (\frac{\partial U}{\partial s} - \kappa W + \eta V)\frac{\partial}{\partial s}. \quad (17)$$

Differentiating (14) with respect to s and using (4), (15) and (17) we obtain the expressions for A and B as

$$A = \frac{\partial W}{\partial s} + \kappa U - \tau V, \quad (18)$$
$$B = \frac{\partial V}{\partial s} + \tau W - \eta U \quad (19)$$

Using (17), compatibility condition for (4) and (15) gives dynamics of curvatures

$$\frac{d\kappa}{dt} = A_s - \eta C - \tau B - (U_s - \kappa W + \eta V)\kappa \quad (20)$$
$$\frac{d\eta}{dt} = \kappa C - B_s - \tau A + (U_s - \kappa W - \eta V)\eta \quad (21)$$
$$\frac{d\tau}{dt} = C_s + \kappa B + \eta A - (U_s - \kappa W + \eta V)\tau. \quad (22)$$

It is noted that (20) – (22) contain soliton equations as special cases by selecting U, W and V as special functions of curvatures.

Finally, differentiating (14) with respect to time gives

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{dU}{dt} - WA - VB\right)\mathbf{t} + \left(\frac{dW}{dt} + UA - VC\right)\mathbf{n} + \left(\frac{dV}{dt} + WC + UB\right)\mathbf{b} \quad (23)$$

Substituting (8) – (13) into (5) and (6), the compatibility conditions for (23) and (5) are

$$\rho Z(\frac{dU}{dt} - WA - VB) = EZ\frac{\partial\sqrt{g}}{\partial\sigma} - \kappa\sqrt{g}Q_1 + \eta\sqrt{g}Q_2, \quad (24)$$
$$\rho Z(\frac{dW}{dt} + UA - VC) = \frac{\partial Q_1}{\partial\sigma} + EZ\kappa\sqrt{g}(\sqrt{g}-1) - \eta\sqrt{g}Q_2, \quad (25)$$
$$\rho Z(\frac{dV}{dt} + WC + UB) = \frac{\partial Q_2}{\partial\sigma} - EZ\eta\sqrt{g}(\sqrt{g}-1) + \eta\sqrt{g}Q_1 \quad (26)$$

by comparing the coefficients of $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$, respectively. Similarly substituting (15) into the left side hand of (6) and comparing the coefficients of $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ to obtain

$$2\rho I\frac{dC}{dt} = GJ\frac{\partial\tau\sqrt{g}}{\partial\sigma} \quad (27)$$
$$\rho I(-\frac{dB}{dt} + AC) = EI\frac{\partial\eta\sqrt{g}}{\partial\sigma} - \sqrt{g}Q_2 + \alpha g\kappa\tau \quad (28)$$
$$\rho I(\frac{dA}{dt} + BC) = EI\frac{\partial\kappa\sqrt{g}}{\partial\sigma} + \sqrt{g}Q_1 - \alpha g\eta\tau \quad (29)$$

where $\alpha = GJ - EI$. As mentioned above, the resultant shears Q_1 and Q_2 are considered to be determined by the equations of balance of momentum (28) and (29). Therefore we neglect the inertia terms in (28) and (29) because of considering of the string which has quite small geometrical moment of inertia and obtain

$$Q_1 = -EI\frac{\partial\kappa\sqrt{g}}{\partial s} + \alpha\sqrt{g}\eta\tau \quad (30)$$
$$Q_2 = EI\frac{\partial\eta\sqrt{g}}{\partial s} + \alpha\sqrt{g}\kappa\tau. \quad (31)$$

When we consider dynamics of the twist, we must take into the effect of the inertia, i.e., an finite area of cross section. Thus we employ (22) and (27) for dynamics of τ. The variable C plays the role of an auxiliary variable in this theory. Therefore, eight equations (16), (20), (21), (22), (24), (25), (26) and (27) for unknown variables $G, \kappa, \eta, \tau, U, W, V$ and C are considered as a new basic equations for the extensible string.

4. Discrete model of a string

In this section, we propose a discrete model of an extensible string discussed in the previous section. The equation for balance of the force is given by

$$m\frac{d^2\mathbf{r}_n}{dt^2} = \mathbf{N}_n - \mathbf{N}_{n-1} - \mathbf{Q}_n + \mathbf{Q}_{n-1} \equiv \mathbf{F}_n \quad (32)$$

where m is mass of a bead, $\mathbf{r}_n$ is a position vector in the space, $\mathbf{N}_n$ and $\mathbf{Q}_n$ is resultant axial force and shear force of a spring, respectively(Fig. 1).

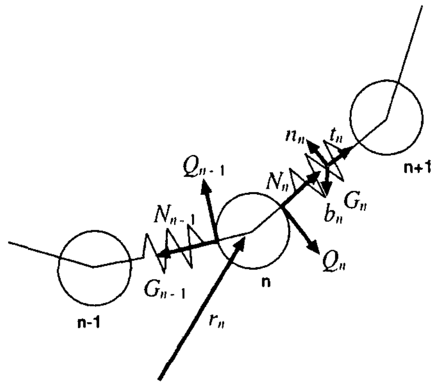

Figure 1: Discrete model of one-dimensional elastic media

The linear constitutive equations for $\mathbf{N}_n$ and $\mathbf{Q}_n$ are given by

$$\mathbf{N}_n = k\frac{|\mathbf{r}_{n+1}-\mathbf{r}_n|-l}{l}\mathbf{t}_n \equiv k\frac{G_n-l}{l}\mathbf{t}_n, \quad (33)$$

$$\mathbf{Q}_n = \frac{\mathbf{M}^l_{n+1}-\mathbf{M}^r_n+\mathbf{R}_n}{G_n}, \quad (34)$$

where

$$\mathbf{M}^r_n = \frac{K}{l}\frac{\alpha_n}{\sin\alpha_n}(-\mathbf{t}_{n-1}+\cos\alpha_n\mathbf{t}_n), \quad (35)$$

$$\mathbf{M}^l_n = \frac{K}{l}\frac{\alpha_n}{\sin\alpha_n}(\mathbf{t}_n-\cos\alpha_n\mathbf{t}_{n-1}), \quad (36)$$

$$\mathbf{R}_n = \mathbf{\Omega}_n\times\mathbf{t}_n, \quad (37)$$

$$\mathbf{\Omega}_n = \frac{h}{l}(\Delta\psi_{n+1}\mathbf{t}_{n+1}-\Delta\psi_n\mathbf{t}_n). \quad (38)$$

$\mathbf{M}^r_n$ and $\mathbf{M}^l_n$ are the bending moment applied to a bead from right and left side, respectively(Fig.2). $\mathbf{\Omega}_n$ is a twist vector of the axis spring(Fig.3) and $\mathbf{R}_n$ is the shear force which comes from twisting. In these equations, k is a spring constant for axis, K a spring constant for bending, l an unstretched length of an axis spring.

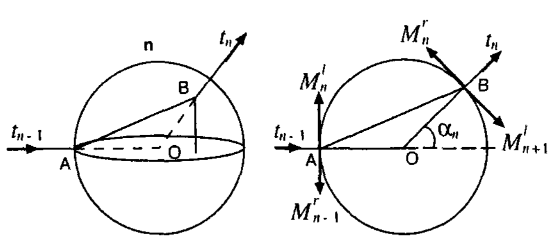

Figure 2: Bending moment in the discrete model

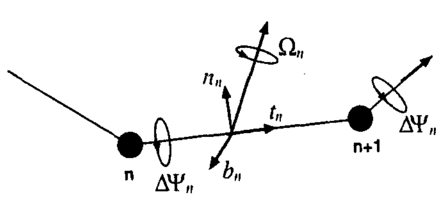

Figure 3: Twist in the discrete model

α_n is defined in Fig.2 and $\Delta\phi_n$ is one of the relative Euler's angles and they relate local orthonormal bases as

$$E_n\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\\ \mathbf{b}_n\end{pmatrix}=\begin{pmatrix}P1_{n+1} & P2_{n+1} & P3_{n+1}\\ P4_{n+1} & P5_{n+1} & P6_{n+1}\\ P7_{n+1} & P8_{n+1} & P9_{n+1}\end{pmatrix}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\\ \mathbf{b}_n\end{pmatrix} \quad (39)$$

where E_n is a shift operator defined by $E_nf(n)=f(n+1)$. $Pi_n(n=1,\cdots,9)$ are expressed by relative Euler's angles $\Delta\psi_n, \Delta\theta_n$ and $\Delta\phi_n$[3]. The equation (39) is considered as a discrete version of (4). We divide (39) by G_n and also taking $G_n\to 0$ with the formula $G_n/l\to\sqrt{g}$ in the limit, from (4) we obtain the relations

$$\kappa = \lim_{G_n\to 0}\frac{\Delta\phi_n}{G_n},$$
$$\eta = \lim_{G_n\to 0}\frac{\Delta\theta_n}{G_n},$$
$$\tau = \lim_{G_n\to 0}\frac{\Delta\psi_n}{G_n}.$$

Next we consider the limit of (33). Following the same procedure above, we obtain

$$\mathbf{N}=k(\sqrt{g}-1)\mathbf{t} \quad (40)$$

This coincides with (10) if $k=EZ$. In the case of (34), we obtain the continuous limit of the shear force $\mathbf{Q}$ as

$$\begin{aligned}\mathbf{Q} &= (K\frac{\partial\kappa\sqrt{g}}{\partial s}+(K-h)\sqrt{g}\tau\eta)\mathbf{n}\\ &\quad +(-K\frac{\partial\eta\sqrt{g}}{\partial s}+(K-h)\sqrt{g}\tau\kappa)\mathbf{b}\\ &= -Q_1\mathbf{n}-Q_2\mathbf{b}.\end{aligned} \quad (41)$$

Thus these also coincide with (30) and (31) if $K=EI$ and $h=GJ$. Putting $m=\rho l$ and dividing (32) by l and using

$$\lim_{l\to 0}\frac{\mathbf{N}_n-\mathbf{N}_{n-1}-\mathbf{Q}_n+\mathbf{Q}_{n-1}}{l}=\frac{\partial\mathbf{N}}{\partial\sigma}-\frac{\partial\mathbf{Q}}{\partial\sigma} \quad (42)$$

it is apparent that (32) becomes (5) in the continuous limit.

Next, we consider dynamics of twisting of the discrete string. Since this discrete model naturally contains the representation of (30) and (31), then we only consider $\mathbf{t}_n$ component of equations for the moment as discussed in the previous section. Such discrete equation is given by(Fig.1.3)

$$\rho I(\mathbf{n}_n\times\frac{d^2\mathbf{n}_n}{dt^2}+\mathbf{b}_n\times\frac{d^2\mathbf{b}_n}{dt^2})_{\mathbf{t}_n}=(\frac{\mathbf{\Omega}_n}{l})_{\mathbf{t}_n}, \quad (43)$$

where picking up $\mathbf{t}_n$ component is represented by $(\cdots)_{\mathbf{t}_n}$. Since

$$\lim_{l\to 0}(\frac{\mathbf{\Omega}_n}{l})_{\mathbf{t}_n}=h\frac{\partial\tau\sqrt{g}}{\partial\sigma}, \quad (44)$$

(43) coincides with $\mathbf{t}_n$ component of (6) in the limit.

5. Discrete soliton theory

Let us introduce the discrete soliton theory in order to analyze the discrete model of a string. We put velocity of a bead and dynamics for the orthonormal basis as

$$\frac{d\mathbf{r_n}}{dt} = U_n\mathbf{t}_n + W_n\mathbf{n}_n + V_n\mathbf{b}_n, \tag{45}$$

$$\frac{d}{dt}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\\ \mathbf{b}_n\end{pmatrix} = \begin{pmatrix}0 & A_n & B_n\\ -A_n & 0 & C_n\\ -B_n & -C_n & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\\ \mathbf{b}_n\end{pmatrix}. \tag{46}$$

Differentiating

$$\mathbf{r}_{n+1} = \mathbf{r}_n + G_n\mathbf{t}_n \tag{47}$$

with respect to t and using (45), (39) and (46), we obtain

$$\frac{dG_n}{dt} = -U_n + P1_{n+1}U_{n+1} + P4_{n+1}W_{n+1} + P7_{n+1}V_{n+1}, \tag{48}$$

$$\begin{aligned} A_n &= \frac{1}{G_n}(-W_n + P2_{n+1}U_{n+1} + P5_{n+1}W_{n+1} \\ &\quad + P8_{n+1}V_{n+1}), \\ B_n &= \frac{1}{G_n}(-V_n + P3_{n+1}U_{n+1} + P6_{n+1}W_{n+1} \\ &\quad + P9_{n+1}V_{n+1}). \end{aligned} \tag{49}$$

From the compatibility condition of (39) and (46) we obtain time evolution equations for relative Euler's angles:

$$\begin{aligned} \frac{d\Delta\phi_n}{dt} &= -A_{n-1} + A_n\frac{\cos\Delta\psi_n}{\cos\Delta\theta_n} - B_n\frac{\sin\Delta\psi_n}{\cos\Delta\theta_n} \\ &\quad + B_{n-1}\sin\Delta\phi_n\tan\Delta\theta_n \\ &\quad - C_{n-1}\tan\Delta\theta_n\cos\Delta\phi_n, \end{aligned} \tag{50}$$

$$\begin{aligned} \frac{d\Delta\theta_n}{dt} &= B_{n-1}\cos\Delta\phi_n - B_n\cos\Delta\psi_n \\ &\quad + C_{n-1}\sin\Delta\phi_n - A_n\sin\Delta\psi_n, \end{aligned} \tag{51}$$

$$\begin{aligned} \frac{d\Delta\psi_n}{dt} &= C_n - C_{n-1}\frac{\cos\Delta\phi_n}{\cos\Delta\theta_n} + B_{n-1}\frac{\sin\Delta\phi_n}{\cos\Delta\theta_n} \\ &\quad - B_n\sin\Delta\psi_n\tan\Delta\theta_n \\ &\quad + A_n\tan\Delta\theta_n\cos\Delta\psi_n \end{aligned} \tag{52}$$

These equations also contain discrete soliton equations as special cases, which are integrable by the soliton theory. These are a discrete version of (20)–(22).

Substituting (38) and (46) into the moment equation (43) we obtain

$$\frac{dC_n}{dt} = \frac{h}{2\rho I l^2}(\Delta\psi_{n+1}\cos\Delta\phi_{n+1}\cos\Delta\theta_{n+1} - \Delta\psi_n). \tag{53}$$

Next, differentiating the velocity (45) with respect to time gives

$$\begin{aligned} \frac{d^2\mathbf{r}_n}{dt^2} &= \left(\frac{dU_n}{dt} - A_nW_n - B_nV_n\right)\mathbf{t}_n \\ &\quad + \left(\frac{dW_n}{dt} + A_nU_n - C_nV_n\right)\mathbf{n}_n \\ &\quad + \left(\frac{dV_n}{dt} + B_nU_n + C_nW_n\right)\mathbf{b}_n. \end{aligned} \tag{54}$$

Comparing coefficients of (54) and (32), we obtain

$$\frac{dU_n}{dt} = A_nW_n + B_nV_n + \frac{(\mathbf{F}_n)_{\mathbf{t}_n}}{m}, \tag{55}$$

$$\frac{dW_n}{dt} = -A_nU_n + C_nV_n + \frac{(\mathbf{F}_n)_{\mathbf{n}_n}}{m}, \tag{56}$$

$$\frac{dV_n}{dt} = -B_nU_n - C_nW_n + \frac{(\mathbf{F}_n)_{\mathbf{b}_n}}{m}. \tag{57}$$

Therefore eight equations (48), (50)–(52), (53), (55)–(57) for $G_n, \Delta\phi, \Delta\theta, \Delta\psi, C_n, U_n, W_n$ and V_n are basic equations for the discrete string. These are a discrete version of the continuous equations given in the previous section.

5. Numerical scheme

The proposed discrete model is suitable for computer simulation of dynamics of a string. We can integrate eight equations with respect to time by some explicit methods, such as the Runge-Kutta method and we did not use matrix calculations. Moreover, these equations converge to the Cosserat string in the $l \to 0$ limit. Thus we can simulate dynamics of an elastic string in a desired accuracy by choosing small l. Figure 4 shows a snapshot of the expanding helix in free space, which is an exact solution of complex-generalized mKdV equation in its initial condition. Other examples, such as pully-belt systems, will be appear elsewhere.

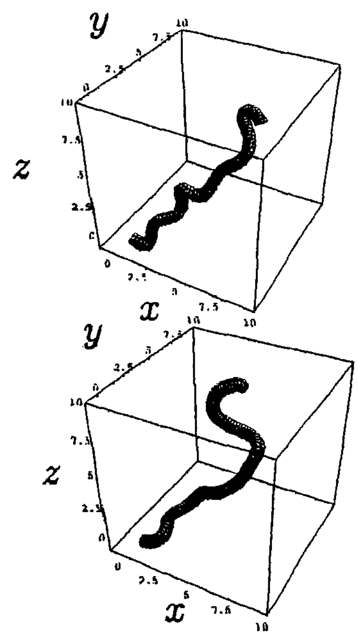

Figure 4: Snapshot of helix motion

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